PROJECTILE MOTION IN POLAR COORDINATES

Nawar Ismail

14th November, 2019

In the simplest case, projectile motion describes the motion of an object that has gravity as the only acting force, resulting in a downward acceleration of $a_y = -g$, where $g \approx 9.8$ m/s^2. Typically there will be some initial velocity $\boldsymbol{v}$, which can be decomposed into $v_x = v\cos(\theta)$ and $v_y = v\sin(\theta)$, where $v = ||\boldsymbol{v}(t)||$. The question is how to model the position of the object. There are standard equations that parametrically calculate the object's coordinates $(x(t), y(t))$ as a function of time. However, in certain cases it may be more useful to do this in polar coordinates $(r(t), \theta(t))$. Like a few of my math problems, I tried this much earlier but didn't have the mathematical *prowess* to solve it. Coming back to it now, it is much easier! We will only need to make use of one (constant-acceleration) kinematic equation for each dimension,

$$r_i = v_i t + \frac{1}{2}a_i t^2, \;\; i \in \{x, y\} \tag{1}$$

where v denotes the initial velocity. Assuming we launch at an initial angle ϕ, we can start with the x dimension which has no acceleration and solve for an unknown to eliminate. Let's choose θ,

$$x = r\cos(\theta) = v\cos(\phi)t \tag{2}$$

$$\implies \theta = \arccos\left(\cos(\phi)\frac{vt}{r}\right). \tag{3}$$

Combining this with the y dimension allows us solve for r,

$$y = r\sin(\theta) = v\sin(\phi)t - \frac{1}{2}gt^2. \tag{4}$$

Noting that $\sin(\arccos(x)) = \sqrt{1-x^2}$ we get,

$$r\sqrt{1-\left(\cos(\phi)\frac{vt}{r}\right)^2} = v\sin(\phi)t - \frac{1}{2}gt^2 \qquad \text{replace } \sin(\theta) \tag{5}$$

$$r^2 - (\cos(\phi)vt)^2 = \left(v\sin(\phi)t - \frac{1}{2}gt^2\right)^2 \qquad \text{squaring} \tag{6}$$

$$r^2 - \cos^2(\phi)v^2t^2 = v^2\sin^2(\phi)t^2 - \frac{1}{2}gv\sin(\phi)t^3 + \frac{1}{2^2}g^2t^4 \qquad \text{expanding} \tag{7}$$

$$r^2 = \cos^2(\phi)v^2t^2 + v^2\sin^2(\phi)t^2 - \frac{1}{2}gv\sin(\phi)t^3 + \frac{1}{2^2}g^2t^4 \qquad \text{rearranging} \tag{8}$$

$$= v^2t^2 - \frac{1}{2}gv\sin(\phi)t^3 + \frac{1}{2^2}g^2t^4 \qquad cos^2 + sin^2 = 1 \tag{9}$$

$$= v^2t^2\left(1 - \frac{1}{2v}gt\sin(\phi) + \frac{1}{v^2 2^2}g^2t^2\right) \qquad \text{factoring} \tag{10}$$

$$\therefore r = vt\sqrt{1-\alpha\sin(\phi)+\alpha^2} \qquad \alpha \equiv gt/2v \tag{11}$$

Now we still need ϕ. Returning to Eq. 2 we find that,

$$r(t) = \frac{\cos(\phi)}{\cos(\theta)}vt \tag{12}$$

$$\tag{13}$$

combining this with Eq. 4 we get,

$$\frac{\cos(\phi)}{\cos(\theta)}\sin(\theta)vt = v\sin(\phi)t - \frac{1}{2}gt^2 \tag{14}$$

$$\frac{\cos(\phi)}{\cos(\theta)}\sin(\theta) = \sin(\phi) - \alpha \tag{15}$$

$$\tan(\theta) = \tan(\phi) - \frac{\alpha}{\cos(\phi)} \tag{16}$$

$$\therefore \theta = \arctan\left(\tan(\phi) - \alpha\sec(\phi)\right) \tag{17}$$

Ultimately we have arrived at:

$$\boxed{\begin{aligned} r(t) &= vt\sqrt{1 - \frac{gt}{2v}\sin(\phi) + \frac{g^2t^2}{4v^2}} && (18)\\ \theta(t) &= \arctan\left(\tan(\phi) - \frac{gt}{2v}\sec(\phi)\right) && (19)\end{aligned}}$$

There are several interesting limits,

1. $$\lim_{t\to\infty} r(t) = \frac{gt^2}{2}$$

   This would be the same as the y coordinate for an object with zero initial velocity. This implies the vertical acceleration will eventually dominate all the motion.

2. $$\lim_{t\to 0} r(t) = vt$$

   This is the same as the x coordinate. This implies the vertical acceleration has very little impact on the distance in this regime.

3. $$\lim_{t\to\infty} \theta(t) = -\pi/2$$

   Again, since the vertical motion dominates at large t this corresponds to the object essentially going straight down.

4. $$\lim_{t\to 0} \theta(t) = \phi$$

   Since the $\phi \in [-\pi/2, \pi/2]$, this limit can be reduced to the angle of launch, ϕ.

5. $$\theta(t) = 0 \tag{20}$$

   $$\tan(\phi) = \frac{gt}{2v\cos(\phi)} \tag{21}$$

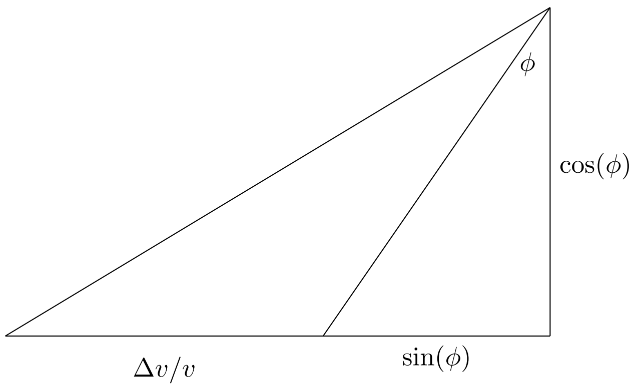
   $$t = \frac{2v}{g}\sin(\phi) \tag{22}$$

   This is the time that the object will hit the ground (assuming it is lanched on the ground). When launched straight up, this occurs at $t = 2v/g$.

We can gain a bit more insight by trying to look at Eq. 11 for a geometric interpretation. Completing the square (with $a = 1, b = -\sin(\phi), c = 1$) yields

$$r = \frac{vt}{2}\sqrt{(2\alpha - \sin(\phi))^2 + \cos^2(\phi) + 3} \tag{23}$$

Looking at the definition for α, we see that it can be interpreted as a change in y-velocity divided by the initial velocity, $\Delta v/v$ (only focusing on the magnitude, and not the sign or direction). Now, if we squint our eyes (and ignore the "3" - which is valid when α grows large), we can see that this looks sort of like a triangle with side length $\cos\phi$, and $\sin\phi + \Delta v/v$.



Here the length $\Delta v/v$ varies with time, while the other lengths remain fixed according to the initial setup. This offers an interesting interpretation, where with a positive Δv, we see that the unitless velocity - the sum of the Δ and sin terms - points positively/upwards (left due to orientation). However, since the vertical acceleration is negative, the Δ term actually should take away from the sin portion. Eventually it will be equal in length, and this results in the (again unitless) velocity being zero. This would correspond to the peak of the motion. As the Δ term increases evermore, the initial velocity becomes negligible (as we can imagine the Δ segment stretching infinitely far to the right).